Continuity estimates for the $p$-Laplace type equation

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1. Existence and uniqueness results - Continuity estimates
2. Stability estimates
3. An application

References

- Giannetti, Greco, Moscariello *Diff. Int. Eq.* (2013)
- Moscariello *To appear*
Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a Lipschitz bounded domain. Consider the problem

\begin{align}
\text{div } A(x, \nabla u) &= \text{div } h \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align}

where $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector field satisfying the following conditions for a. e. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^N$

\begin{align*}
A(x, 0) &= 0 \\
\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle &\geq a |\xi - \eta|^2 (|\xi| + |\eta|)^{p-2} \\
|A(x, \xi) - A(x, \eta)| &\leq b |\xi - \eta| (|\xi| + |\eta|)^{p-2}
\end{align*}

where $p > 1$, $0 < a \leq b$. 
\( h = (h^1, h^2, \ldots, h^N) \) is a vector field in \( L^s(\Omega, \mathbb{R}^N), 1 \leq s \leq q \), \( pq = p + q \).

If \( \max\{1, p - 1\} \leq r \leq p \) we say that

**Definition 1**

A function \( u \in W^{1, r}_0(\Omega) \) is a solution of (1) if

\[
\int_\Omega \langle A(x, \nabla u), \nabla \varphi \rangle \, dx = \int_\Omega \langle f, \nabla \varphi \rangle \, dx,
\]

for every \( \varphi \in C_0^\infty(\Omega) \).

If \( s \geq \frac{r}{p - 1} \) we can consider \( \varphi \in W^{1, \frac{r}{r-p+1}}(\Omega) \) with compact support.

For \( r < p \) such a solution may have ”infinite energy”.
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for every $\varphi \in C_0^\infty(\Omega)$.

If $s \geq r/(p - 1)$ we can consider $\varphi \in W^{1,\frac{r}{r-p+1}}(\Omega)$ with compact support.

For $r < p$ such a solution may have "infinite energy".
1 Existence and uniqueness

- $r = p$
  
  If $h \in L^q(\Omega, \mathbb{R}^N)$, $\exists! u \in W_0^{1,p}(\Omega)$ that solves (1).
  
  Leray-Lions 65, Browder 70.

- $r < p$, ”Existence”
  
  If $\text{div } h = f \in L^\gamma(\Omega)$, $\gamma \geq 1$
  
  $\exists u \in W_0^{1,1}(\Omega)$ that solves (1).

  Stampacchia 65, Brezis-Strauss 73.
  
  Boccardo–Galloüet 89 ($p > 2 - \frac{1}{N}$, $\gamma = 1$)
  
  Boccardo–Galloüet 2012 ($1 < p < 2 - \frac{1}{N}$, $\gamma = \frac{N}{Np-N+1}$)
  
  Murat 94, Alvino–Mercaldo 2008

  SOLA (solution obtained by approximation)

  Regularity results for such a solution have been developed by Mingione (2007,2010)
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Regularity results for such a solution have been developed by Mingione (2007,2010)
"\[ \exists r_0 < p \text{ s.t. if } h \in L^{\frac{r}{p-1}}(\Omega, \mathbb{R}^N), \ r_0 \leq r < p \text{ then there exists } u \in W^{1,r}_0(\Omega) \text{ that solves (1)} \]

\[ p = 2 \text{ Boccardo 97, Fiorenza–Sbordone 98} \]
\[ p \neq 2 \text{ Iwaniec–Sbordone 2001} \]
$r < p$ "uniqueness"

**SOLA solutions are unique**, i.e.

"different approximating problems of (1) have the same limit solutions $u$"

(Dall'Aglio 96, Boccardo 97, Boccardo–Galloüet 2012)

For "distributional" solutions the uniqueness generally fails when $r$ is far form $p$

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If $r$ is close to $p$, at the present time the problem is unclear unless for

$$p = 2$$

"$\exists r_0 < 2 < r_1$ s.t. if $h \in L^r(\Omega, \mathbb{R}^N)$, $r_0 < r < r_1$, $\exists! u \in W^{1,r}_0(\Omega)$ that solves (1)"

$$\|\nabla u\|_{L^r} \leq c \|h\|_{L^r}.$$ 

Previous result can be extended to operators

$$\mathcal{A}(x, \xi) \simeq b(x)\xi$$

with $b(x) \in BMO$. (Carozza-Moscariello-Passarelli di Napoli 2002).
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Uniqueness holds in spaces not too much larger than $W_0^{1,p}$

- $|\nabla u|$ in Grand-Lebesgue spaces (Greco–Iwaniec–Sbordone 97) i.e.
  \[ u \in \cap_{0 \leq \varepsilon \leq p-1} W_0^{1,p-\varepsilon}(\Omega) \]

  and
  \[ \|\nabla u\|_{L^p}(\Omega) = \sup_{0 < \varepsilon \leq p-1} \left[ \varepsilon \int_\Omega |\nabla u|^{p-\varepsilon} \, dx \right]^{\frac{1}{p-\varepsilon}} < \infty \]

- $|\nabla u|$ in weak–$L^N$ (Dolzmann–Hungerbühler–Müller 2000) i.e.
  \[ \|\nabla u\|_{L^N,\infty}(\Omega) = \sup_{t>0} t \left| \{ x \in \Omega : |\nabla u| > t \} \right|^{\frac{1}{N}} < +\infty \]

  \[ L^{N,\infty}(\Omega) \sim \text{weak–}L^N \]

Here we consider solutions of (1) with $|\nabla u| \in L^p \log^{-\alpha} L(\Omega)$, $\alpha > 0$, $p > 1$. 

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$L^p \log^{-\alpha} L(\Omega)$, \quad $\alpha > 0$, \quad $p > 1$

This is the Orlicz space generated by the function

$$\Phi(t) = t^p \log^{-\alpha}(a + t), \quad t \geq 0,$$

and $a \geq e$.

\[ f : \Omega \to \mathbb{R} \]

\[ f \in L^p \log^{-\alpha} L(\Omega) \iff \int_{\Omega} |f|^p \log^{-\alpha}(a + |f|) \, dx < \infty. \]

The Luxemburg norm ($\simeq$ Zygmund norm)

$$\|f\|_{L^p \log^{-\alpha} L} = \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi \left( \frac{|f|}{\lambda} \right) \, dx \leq 1 \right\},$$

and $L^p \log^{-\alpha} L(\Omega)$ is a Banach space.

$$\lim_{\varepsilon \to 0} \varepsilon^{\alpha/p} \|f\|_{p-\varepsilon} = 0$$
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\[ \lim_{\varepsilon \to 0} \varepsilon^{\frac{\alpha}{p}} \| f \|_{p-\varepsilon} = 0 \]
\[ 0 < \alpha \leq 1 \quad L^p \log^{-\alpha} L \not\subset \text{weak–}L^p \quad \text{(Greco '93)} \]

\[
\text{weak–}L^p \not\subset L^p \log^{-\alpha} L
\]

\[ \alpha > 1 \quad \text{weak–}L^p \subset L^p \subset L^p \log^{-\alpha} L; \]
Then we can prove the following

**Theorem 1**

Let $1 < p < \infty$, $p \neq 2$. For each $h \in L^q \log^{-\alpha} L(\Omega, \mathbb{R}^N)$, $0 < \alpha \leq \frac{p}{|p-2|}$, problem (1) admits a unique solution s.t. $|\nabla u| \in L^p \log^{-\alpha} L(\Omega)$. Moreover,

$$
\|\nabla u\|^p_{L^p \log^{-\alpha} L} \leq C \|h\|^p_{L^q \log^{-\alpha} L}
$$

where $C = C(N, p, \alpha, a, b)$. 
For $0 < \alpha \leq \frac{p}{|p-2|}$ the operator

$$\mathcal{H} : L^q \log^{-\alpha} L(\Omega, \mathbb{R}^N) \to L^p \log^{-\alpha} L(\Omega, \mathbb{R}^N)$$

which carries $h$ into $\nabla u$ is well defined.

- $\mathcal{H}$ is continuous
- $\mathcal{H}$ is uniformly continuous when $0 < \alpha < \frac{p}{|p-2|}$

$$\|\mathcal{H}u - \mathcal{H}v\|_{L^p \log^{-\alpha} L} \leq C \|h - g\|_{L^q \log^{-\alpha} L}$$

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Remark 1

For $1 < \alpha \leq \frac{p}{|p-2|}$, Theorem 1 improves the results of Greco-Iwaniec-Sbordone (97), since in this case

$$L^p) \subset L^p \log^{-\alpha} L$$

Remark 2

An improvement of Theorem 1 has been recently proved by F.Farroni (to appear)

$$|\nabla u| \in L^p \log^{-\alpha} L (\log \log L)^{-\beta}$$
Remark 1
For $1 < \alpha \leq \frac{p}{|p-2|}$, Theorem 1 improves the results of Greco-Lwaniec-Sbordone (97), since in this case

$$L^p(\log L)^{\frac{14}{51}} \subset L^p \log^{-\alpha} L$$

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$$|\nabla u| \in L^p \log^{-\alpha} L (\log \log L)^{-\beta}$$
For \( p=2 \) as a consequence of the estimates in \( L^{2 \pm \epsilon} \) and the interpolation theorem of Bennett-Rudnick we get

\[
\| \nabla u \|_{L^2 \log^{-\alpha} L} \leq c \| h \|_{L^2 \log^{-\alpha} L}
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for any \(-\infty < \alpha < +\infty\).

Anyway a situation similar to the case \( p \neq 2 \) occurs if we consider the problem
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Anyway a situation similar to the case $p \neq 2$ occurs if we consider the problem
\[
\begin{aligned}
(1.1) \quad \begin{cases}
\text{div}(A(x, \nabla u) + B(x)u) = \text{div} h & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\end{aligned}
\]

where \( B(x) \in \text{weak-}L^N(\Omega, \mathbb{R}^N) \)

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\int_{\Omega} \langle A(x, \nabla u) + B(x)u, \nabla \varphi \rangle \, dx = \int_{\Omega} h\nabla \varphi \, dx
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for any \( \varphi \in C_0^\infty(\Omega) \).

If \( u \in W^{1,2}_0(\Omega) \) then

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\|u\|_{2^*,2} \leq S_2 \|\nabla u\|_2
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If \( u \in W^{1,2}_0(\Omega) \) then

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\]
The main difficulty in solving problem (1.1) is due to the noncoercivity of the operator

\[ \langle \mathcal{A}u, v \rangle = \int_{\Omega} \langle A(x, \nabla u) + B(x)u, \nabla v \rangle \, dx \]

\[ u, v \in \mathcal{W}^{1,2}_0. \]

(Alvino-Trombetti '82; Boccardo 2009; Droniou 2002)

**Theorem 1.1 (M.)**

Let \( B(x) \in \text{weak}-L^N(\Omega, \mathbb{R}^N) \). Then there exists at most one solution of (1.1) s.t. \( |\nabla u| \in L^2 \log^{-\alpha} L, \quad 0 \leq \alpha \leq 2. \)
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\[ u, v \in W_0^{1,2}. \]

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For the existence of a solution we assume that

\[ \text{dist}_{L^N, L^\infty}(B, L^\infty) < \frac{a}{4S_2} \]  

(1.2)

where \( S_2 \) is the Sobolev constant, and \( a \) is the coercivity constant of \( A(x, \xi) \).

**Theorem 1.2 (M.)**

Assume (1.2). Then, for every \( h \in L^2 \log^{-\alpha} L \), \( 0 \leq \alpha \leq 2 \), problem (1.1) admits a unique solution s.t. \( |\nabla u| \in L^2 \log^{-\alpha} L \). Moreover

\[ \| \nabla u \|_{L^2 \log^{-\alpha} L} \leq c(\| h \|_{L^2 \log^{-\alpha} L} + \| B \|_{L^N, L^\infty}), \]

\( c = c(N, \alpha, a, b) \).
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**Theorem 1.2 (M.)**

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$$\|\nabla u\|_{L^2 \log^{-\alpha} L} \leq c(\|h\|_{L^2 \log^{-\alpha} L} + \|B\|_{L^N,L^\infty}),$$

$$c = c(N,\alpha,a,b).$$
We remark that $L^\infty$ is not dense in weak-$L^N$.

$$\text{dist}_{L^N,\infty}(f, L^\infty) = \inf_{g \in L^\infty} \|f - g\|_{N,\infty}$$

A formula for the distance is the following

$$\text{dist}_{L^N,\infty}(f, L^\infty) = \lim_{k \to \infty} \|f - T_k f\|_{N,\infty}, \quad (1.3)$$

where $T_k f$ is the truncation of $f$ at the level $k$.

Formula (1.3) is due to Carozza - Sbordone (1997).

Remark

Condition (1.2) does not give any smallness control on the norm of $B$ in weak-$L^N$.

For $f(x) = |x|^{-1}$, $\|f - T_k f\|_{L^N,\infty} = \omega_{1/N}^{1/N}$.
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For $f(x) = |x|^{-1}$, $\|f - T_k f\|_{L^N,\infty} = \omega_N^{1/N}$
2. Stability results
Our aim is to ”estimate” the distance between solutions \( u \) and \( v \) of the problem

\[
\begin{align*}
\text{div } A(x, \nabla u) &= \text{div } (|\nabla v|^{p-2} \nabla v) \quad \text{in } \Omega, \\
u &= v \quad \text{on } \partial \Omega,
\end{align*}
\]

when the operator \( A(x, \xi) \) is close to the \( p \)-Laplacian, in the sense that there exists a constant \( M > 0 \) such that

\[
|A(x, \xi) - |\xi|^{p-2}\xi| \leq M|\xi|^{p-1}
\]

for all \( \xi \in \mathbb{R}^N \), for a.e. \( x \in \Omega \).
When $\mathcal{A}(x, \xi)$ is a $p$–harmonic operator, i.e.

$$\mathcal{A}(x, \xi) = \langle A(x)\xi, \xi \rangle^{\frac{p-2}{2}} A(x)\xi$$

with $A : \Omega \to \mathbb{R}^{N \times N}$ a measurable, symmetric matrix field, verifying the ellipticity bounds

$$\mu |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \nu |\xi|^2$$

for all $\xi \in \mathbb{R}^N$, for a.e. $x \in \Omega$, $0 < \mu \leq \nu$, condition (3) is satisfied in terms of $\mu$ and $\nu$. 
Anyway, when $p \geq 2$, following the theory of quasiregular mappings, we use the pointwise inequality

$$|A(x, \xi) - |\xi|^{p-2}\xi| \leq (K_A - 1)|\xi|^{p-1}$$

(4)

that holds true for all $\xi \in \mathbb{R}^N$, for a.e. $x \in \Omega$, with $K_A$ defined as the characteristic of the matrix $A = A(x)$

$$K_A = \text{ess sup}_{x \in \Omega} (1 + |A(x) - I|)^{\frac{p}{2}} .$$

(Iwaniec '83) $K_A \geq 1$; $K_A = 1$ iff $A \equiv I$

$K_A - 1 \simeq "distance measure"$ between the operators
For simplicity we assume \( p \geq 2 \), \( A(x, \xi) \) \( p \)-harmonic operator.

- The case \( r = p \)

If \( u, v \in W^{1,p}(\Omega) \) solve (2) then

\[
\int_{\Omega} \left\langle A(x, \nabla u) - A(x, \nabla v), \nabla u - \nabla v \right\rangle \, dx
\]

\[
= \int_{\Omega} \left\langle |\nabla v|^{p-2} \nabla v - A(x, \nabla v), \nabla u - \nabla v \right\rangle \, dx
\]
Estimate (4) + monotonicity of $A(x, \xi)$

\[ \int_{\Omega} |\nabla u - \nabla v|^p dx \leq c(K_A - 1) \int_{\Omega} |\nabla v|^{p-1} |\nabla u - \nabla v| dx \]

\[ \leq c(K_A - 1) \|\nabla v\|_p^{p-1} \|\nabla u - \nabla v\|_p \]

$c = c(\mu)$. Then, we can conclude

**Proposition 2.1**

\[ \|\nabla u - \nabla v\|_p \leq c(K_A - 1)^{\frac{1}{p-1}} \|\nabla v\|_p \]
Estimate (4) + monotonicity of $\mathcal{A}(x, \xi)$

\[\downarrow\]

\[
\int_{\Omega} |\nabla u - \nabla v|^p \, dx \leq c(K_A - 1) \int_{\Omega} |\nabla v|^{p-1} |\nabla u - \nabla v| \, dx
\]

\[\leq c(K_A - 1) \|\nabla v\|_p^{p-1} \|\nabla u - \nabla v\|_p\]

c = c(\mu). Then, we can conclude

**Proposition 2.1**

\[\|\nabla u - \nabla v\|_p \leq c(K_A - 1)^{\frac{1}{p-1}} \|\nabla v\|_p\]
Theorem 2.2

If \( u, v \in W^{1,1}(\Omega) \), with \( | \nabla u|, | \nabla v| \in L^p \log^{-\alpha} L(\Omega) \), \( 0 < \alpha < \frac{p}{p-2} \), solve (2) then

\[
\| \nabla u - \nabla v \|_{L^p \log^{-\alpha} L(\Omega)} \leq CK \frac{1-\gamma}{p-1} (K_A - 1)^{\frac{\gamma}{p-1}} \| \nabla v \|_{L^p \log^{-\alpha} L(\Omega)}
\]

where \( \gamma = 1 - \alpha \frac{p-2}{p} \) and \( C = C(N, p, \alpha, \mu, \nu) \).
We don’t know if previous theorem holds for

\[ \alpha = \frac{p}{p - 2} \]

even if the uniqueness of (1) holds true in this case.

In the general case, assuming condition (3), a similar result holds true with a more hard calculation.
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In the general case, assuming condition (3), a similar result holds true with a more hard calculation.
In order to prove previous results the main tools are:

- The ’stability’ of the Hodge decomposition [Iwaniec-Sbordone]

\[ |\nabla u|^{r-p}\nabla u = \nabla \phi + H \]

\[ \phi \in W^{1, \frac{r}{r-p+1}}_0(\Omega), \quad H \in L^{\frac{r}{r-p+1}}(\Omega, \mathbb{R}^N), \quad \text{div } H = 0 \]

- To consider in \( L^p \log^{-\alpha} L \) a norm equivalent to the Luxemburg one
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- To consider in \(L^p \log^{-\alpha} L\) a norm equivalent to the Luxemburg one
Lemma 2.3 (Edmunds-Triebel; Farroni-Greco-M.)

\[ f \in L^p \log^{-\alpha} L(\Omega) \text{ iff } \exists \varepsilon_0 \in (0, p - 1) \text{ s.t.} \]

\[ [f]_{p,\alpha} = \left( \int_0^{\varepsilon_0} \varepsilon^{\alpha-1} \|f\|_{p-\varepsilon}^p d\varepsilon \right)^{\frac{1}{p}} < \infty. \]

Moreover \[ [f]_{p,\alpha} \asymp \|f\|_{L^p \log^{-\alpha} L} \]

This norm just involves the norms of \( f \) in \( L^r, r < p \).
Proof of Theorem 2.2  

Assume that $u$ and $v$ solve the problem

\[
\begin{align*}
(P) \begin{cases}
\text{div } A(x, \nabla u) &= \text{div } (|\nabla v|^{p-2} \nabla v) \quad \text{in } \Omega, \\
u &= v \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}
\]

s.t. $|\nabla u|, |\nabla v| \in L^p \log^{-\alpha} L(\Omega)$.

Then $|\nabla u|, |\nabla v| \in L^r(\Omega)$ for any $1 < r < p$. For $r = p - \varepsilon p$, $0 < \varepsilon < \varepsilon_p = p - r_0$, we may use the Hodge Decomposition of the vector field

\[
|\nabla u - \nabla v|^{-\varepsilon p}(\nabla u - \nabla v) = \nabla \phi + H
\]

with

\[
\phi \in \mathcal{W}^{1, \frac{p-\varepsilon p}{1-\varepsilon p}}_0(\Omega), \quad \text{and} \quad H \in L^{\frac{p-\varepsilon p}{1-\varepsilon p}}(\Omega) \quad \text{s.t. div } H = 0.
\]
Moreover the following estimates hold (Iwaniec-Sbordone 92-94):

(i) \[ \| \nabla \phi \|_{p-\varepsilon p \over 1-\varepsilon p} \leq C(N, p) \| \nabla u - \nabla v \|_{p-\varepsilon p}^{1-\varepsilon p} \]

(ii) \[ \| H \|_{p-\varepsilon p \over 1-\varepsilon p} \leq C(N, p) \varepsilon \| \nabla u - \nabla v \|_{p-\varepsilon p}^{1-\varepsilon p}. \]

By definition of solution, we are legitimate to use \( \phi \) as test function in \((P)\)

\[
\int_{\Omega} \langle A(x, \nabla u) - A(x, \nabla v), \nabla \phi \rangle \, dx \\
= \int_{\Omega} \langle |\nabla v|^{p-2} \nabla v - A(x, \nabla v), \nabla \phi \rangle \, dx.
\]
Moreover the following estimates hold (Iwaniec-Sbordone 92-94):

(i) \( \| \nabla \phi \|_{p-\epsilon p}^{\frac{1}{1-\epsilon p}} \leq C(N, p) \| \nabla u - \nabla v \|_{p-\epsilon p}^{1-\epsilon p} \)

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By definition of solution, we are legitimate to use \( \phi \) as test function in \( (P) \)

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\int_{\Omega} \langle A(x, \nabla u) - A(x, \nabla v), \nabla \phi \rangle \, dx \\
= \int_{\Omega} \langle |\nabla v|^{p-2} \nabla v - A(x, \nabla v), \nabla \phi \rangle \, dx.
\]
Then, by the coercivity we have

\[ \| \nabla u - \nabla v \|_{p-\varepsilon p}^{p-\varepsilon p} \leq \int_\Omega \langle A(x, \nabla u) - A(x, \nabla v), (\nabla u - \nabla v)|\nabla u - \nabla v|^{\varepsilon p} \rangle \, dx \]

\[ \leq \frac{C(N, p)}{a} \left\{ \int_\Omega \langle |\nabla v|^{p-2}|\nabla v| - A(x, \nabla v), \nabla \phi \rangle \, dx \right\} \]

\[ + \int_\Omega |A(x, \nabla u) - A(x, \nabla v)||H| \, dx \]
From (4) and (ii) with the aid of Hölder inequality we get the following estimate:

\[ \| \nabla u - \nabla v \|_{p-\varepsilon p}^p \leq C \left[ (K_A - 1)^{\frac{p}{p-1}} \| \nabla v \|_{p-\varepsilon p}^p + \varepsilon^{\frac{p}{p-2}} \| \nabla |u| + |\nabla v| \|_{p-\varepsilon p}^p \right] \]

that implies for \( \varepsilon_p \) sufficiently small

\[ \| \nabla u - \nabla v \|_{p-\varepsilon p}^p \leq C \left[ (K_A - 1)^{\frac{p}{p-1}} + \varepsilon^{\frac{p}{p-2}} \right] \| \nabla v \|_{p-\varepsilon p}^p \]

for \( 0 < \varepsilon < \varepsilon_p \) and \( C = C(p, N, \mu, \nu) \).

We multiply by \( \varepsilon^{\alpha-1} \) and integrate with respect to \( \varepsilon \) on \( (0, \vartheta \varepsilon_p), \vartheta \in (0, 1] \)
From (4) and (ii) with the aid of Hölder inequality we get the following estimate:

\[
\| \nabla u - \nabla v \|^p_{p-\varepsilon p} \leq C \left[ (K_A - 1)^{\frac{p}{p-1}} \| \nabla v \|^p_{p-\varepsilon p} + \varepsilon^{\frac{p}{p-2}} \| \nabla u + |\nabla v| \|^p_{p-\varepsilon p} \right]
\]

that implies for \( \varepsilon_p \) sufficiently small

\[
\| \nabla u - \nabla v \|^p_{p-\varepsilon p} \leq C \left[ (K_A - 1)^{\frac{p}{p-1}} + \varepsilon^{\frac{p}{p-2}} \right] \| \nabla v \|^p_{p-\varepsilon p}
\]

for \( 0 < \varepsilon < \varepsilon_p \) and \( C = C(p, N, \mu, \nu) \).

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for \( 0 < \varepsilon < \varepsilon_p \) and \( C = C(p, N, \mu, \nu) \).

We multiply by \( \varepsilon^{\alpha-1} \) and integrate with respect to \( \varepsilon \) on \( (0, \vartheta \varepsilon_p), \vartheta \in (0, 1] \)
\[
\int_0^{\vartheta \varepsilon_p} \varepsilon^{\alpha - 1} \| \nabla u - \nabla v \|^p_{p - \varepsilon_p} \, d\varepsilon \leq C \left[ (K_A - 1)^{\frac{p}{p-1}} \int_0^{\vartheta \varepsilon_p} \varepsilon^{\alpha - 1} \| \nabla v \|^p_{p - \varepsilon_p} \, d\varepsilon \\
+ \int_0^{\vartheta \varepsilon_p} \varepsilon^{\frac{p}{p-2} + \alpha - 1} \| \nabla v \|^p_{p - \varepsilon_p} \, d\varepsilon \right] \\
\leq C \left[ (K_A - 1)^{\frac{p}{p-1}} \frac{1}{p^\alpha} \int_0^{\varepsilon_0} \delta^{\alpha - 1} \| \nabla v \|^p_{p - \delta} \, d\delta \\
+ \frac{\vartheta^{\frac{p}{p-2} + \alpha} \varepsilon_p^{\frac{p}{p-2}}}{p^\alpha} \int_0^{\varepsilon_0} \delta^{\alpha - 1} \| \nabla v \|^p_{p - \delta} \, d\delta \right]
\]

where \( \varepsilon_0 = p \varepsilon_p, \delta = \frac{\varepsilon_p}{p - 1}. \)
On the other hand, since for $\tau = \frac{\varepsilon p}{\theta} \geq \varepsilon p$
\[
\|\nabla u - \nabla v\|_{p-\varepsilon p}^p \geq \|\nabla u - \nabla v\|_{p-\tau}^p
\]
we have that the integral in the left hand side
\[
\int_0^{\vartheta \varepsilon} \varepsilon^{\alpha - 1}\|\nabla u - \nabla v\|_{p-\varepsilon p}^p \, d\varepsilon \geq \left(\frac{\vartheta}{p}\right)^\alpha \int_0^{\varepsilon_0} \tau^{\alpha - 1}\|\nabla u - \nabla v\|_{p-\tau}^p \, d\tau
\]
Then by Lemma 2.3 we get that
\[
\vartheta^\alpha \|\nabla u - \nabla v\|_{L^p \log^{-\alpha} L}^p 
\leq C \left\{ (K_A - 1)^{\frac{p}{p-1}} \|\nabla v\|_{L^p \log^{-\alpha} L}^p + \vartheta^{\frac{p}{p-2}} \|\nabla v\|_{L^p \log^{-\alpha} L}^p \right\}
\]
The conclusion follows by choosing
\[
\vartheta^{\frac{p}{p-2}} = \left(\frac{K_A - 1}{K_A}\right)^{\frac{p}{p-1}}.
\]
On the other hand, since for $\tau = \frac{\varepsilon p}{\theta} \geq \varepsilon p$

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we have that the integral in the left hand side

$$\int_{0}^{\varepsilon p} \varepsilon^{\alpha-1} \|\nabla u - \nabla v\|_{p-\varepsilon p}^{p} \, d\varepsilon \geq \left(\frac{\vartheta}{p}\right)^{\alpha} \int_{0}^{\varepsilon 0} \tau^{\alpha-1} \|\nabla u - \nabla v\|_{p-\tau}^{p} \, d\tau$$

Then by Lemma 2.3 we get that

$$\vartheta^{\alpha} \|\nabla u - \nabla v\|_{L^{p} \log^{-\alpha} L}^{p} \leq C \left\{ (K_{A} - 1)^{\frac{p}{p-1}} \|\nabla v\|_{L^{p} \log^{-\alpha} L}^{p} + \vartheta^{\frac{p}{p-2}} \|\nabla v\|_{L^{p} \log^{-\alpha} L}^{p} \right\}$$

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The conclusion follows by choosing

$$\vartheta^{\frac{p}{p-2}} = \left( \frac{K_A - 1}{K_A} \right)^{\frac{p}{p-1}}.$$
3. An application

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded open set.

If $f : \Omega \rightarrow \mathbb{R}^N$ is a map in $W^{1,N-1}_{loc}(\Omega, \mathbb{R}^N)$ s.t. $|\text{adj } Df| \in L^{\frac{N}{N-1}}_{loc}(\Omega)$ then

$$N\omega_N^{\frac{1}{N}} \left| \int_{B_r(x_0)} J_f(x) \, dx \right|^{\frac{N-1}{N}} \leq \int_{\partial B_r(x_0)} |\text{adj } Df| \, d\mathcal{H}^{N-1}$$

for each ball $B_r(x_0) \subset \subset \Omega$ and for a.e. $x_0 \in \Omega$ and a.e. $r$.

Here $\text{adj } Df = (\text{cof } Df)^t$, $J_f = \det Df$.

$\omega_N$ is the measure of the unit ball of $\mathbb{R}^N$. 
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Let \( \Omega \subset \mathbb{R}^N, N \geq 2 \), be a bounded open set.

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\]
then
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N \omega_N^{\frac{1}{N}} \left| \int_{B_r(x_0)} J_f(x) \, dx \right|^{\frac{N-1}{N}} \leq \int_{\partial B_r(x_0)} |\text{adj } Df| \, d\mathcal{H}^{N-1}
\] (5)

for each ball \( B_r(x_0) \subset \subset \Omega \) and for a.e. \( x_0 \in \Omega \) and a.e. \( r \).

Here \( \text{adj } Df = (\text{cof } Df)^t \) \( J_f = \det Df \)

\( \omega_N \) is the measure of the unit ball of \( \mathbb{R}^N \).
Inequality (5) is known as the "the integral form" of the isoperimetric inequality.

- Müller-Qi-Yan (94)
- Farroni–M. (2014) best constant in (5)

The isoperimetric inequality states that

$$N \omega_1 \frac{1}{N} |E|^{\frac{N}{N-1}} \leq \mathcal{P}(E)$$

for every Borel set $E$ of $\mathbb{R}^N$.

Quantitative versions by Fuglede 89, Fusco–Maggi–Pratelli ’08, Figalli–Maggi–Pratelli ’10

Assumptions on $f$ and Hadamard’s inequality

$$|J_f| \leq |\text{adj } Df|^{\frac{N}{N-1}} \leq |Df|^N$$

imply that

$$J_f \in L^1_{\text{loc}}(\Omega)$$

$$J_f \in \mathcal{H}^1_{\text{loc}}(\Omega) \quad (\text{Iwaniec–Onninen 2002})$$
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The case of equality in (5)

Assume that equality occurs in (5), i.e.

\[
\left| \int_{B_r(x_0)} J_f(x) \, dx \right|^{\frac{N-1}{N}} = \int_{\partial B_r(x_0)} |\text{adj} \, Df| \, d\mathcal{H}^{N-1}
\]

for each ball \( B_r(x_0) \subset \subset \Omega \), for a.e. \( x_0 \in \Omega \) and for a.e. \( r \).

**Theorem 3.1 (Farroni–M.)**

Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain. Assume that \( f : \Omega \to \mathbb{R}^N \) is a continuous, one-to-one mapping satisfying (6); then

either \( J_f \geq 0 \) a.e. in \( \Omega \) or \( J_f \leq 0 \) a.e. in \( \Omega \).

Main tools for the proof are the area formula and Sobolev inequality for BV functions.
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Main tools for the proof are the area formula and Sobolev inequality for BV functions.
If $N \geq 3$ and $f$ is a homeomorphism satisfying (6) then $f$ is the restriction to $\Omega$ of a Möbius transform of $\overline{\mathbb{R}}^N$. More precisely, $f \in W^{1,N}_{loc}(\Omega, \mathbb{R}^N)$ and has the form

$$f(x) = b + \frac{\lambda A(x - a)}{|x - a|^{\alpha}},$$

where $a \in \mathbb{R}^N \setminus \Omega$, $b \in \mathbb{R}^N$, $\lambda \in \mathbb{R}$, $A$ is an orthogonal matrix and $\alpha$ is either 0 or 2.

For $N = 2$ $f$ is harmonic.

As a consequence we get the following

**Proposition 3.2**

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 2$, and let $f \in W^{1,N-1}(\Omega, \mathbb{R}^N)$ be a homeomorphism. Assume that (6) holds and $f = Id$ on $\partial \Omega$. Then

$$f = Id \quad \text{in } \Omega$$
If $N \geq 3$ and $f$ is a homeomorphism satisfying (6) then $f$ is the restriction to $\Omega$ of a Möbius transform of $\mathbb{R}^N$. More precisely, $f \in W_{loc}^{1,N}(\Omega, \mathbb{R}^N)$ and has the form

$$f(x) = b + \frac{\lambda A(x - a)}{|x - a|^{\alpha}},$$

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$$f = Id \quad \text{in } \Omega$$
In view of inequality (5) a natural question is to "measure the distance" between a mapping $f \in W^{1,N-1}_{1oc}(\Omega, \mathbb{R}^N)$, with $J_f \geq 0$ a.e. in $\Omega$, satisfying "equality" (6) and a mapping $g \in W^{1,N-1}_{1oc}(\Omega, \mathbb{R}^N)$, with $J_g \geq 0$ a.e. in $\Omega$, whenever $\exists M \geq 1$ s.t.

$$\int_{\partial B_r(x_0)} |\text{adj } Dg| \, d\mathcal{H}^{N-1} \leq M \left(\int_{B_r(x_0)} J_g(x) \, dx\right)^{\frac{N-1}{N}} \quad (7)$$

for each ball $B_r(x_0) \subset \subset \Omega$, for a.e. $x_0 \in \Omega$.

Inequality (7) can be considered as a quantitative version of (5).
In view of inequality (5) a natural question is to ”measure the distance” between a mapping \( f \in W_{1,\text{loc}}^{1,N-1}(\Omega, \mathbb{R}^N) \), with \( J_f \geq 0 \) a.e. in \( \Omega \), satisfying ”equality” (6) and a mapping \( g \in W_{1,\text{loc}}^{1,N-1}(\Omega, \mathbb{R}^N) \), with \( J_g \geq 0 \) a.e. in \( \Omega \), whenever \( \exists M \geq 1 \) s.t.

\[
\int_{\partial B_r(x_0)} |\text{adj} \ Dg| \ d\mathcal{H}^{N-1} \leq M \left( \int_{B_r(x_0)} J_g(x) \, dx \right)^{\frac{N-1}{N}} \tag{7}
\]

for each ball \( B_r(x_0) \subset \subset \Omega \), for a.e. \( x_0 \in \Omega \).

Inequality (7) can be considered as a quantitative version of (5).
In view of inequality (5) a natural question is to ”measure the distance” between a mapping $f \in W^{1,N-1}_{loc}(\Omega, \mathbb{R}^N)$, with $J_f \geq 0$ a.e. in $\Omega$, satisfying ”equality” (6) and a mapping $g \in W^{1,N-1}_{loc}(\Omega, \mathbb{R}^N)$, with $J_g \geq 0$ a.e. in $\Omega$, whenever $\exists M \geq 1$ s.t.

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Inequality (7) can be considered as a quantitative version of (5).
"Measure the distance"

\[ M \sim 1 \quad \implies \quad \| Df - Dg \| \sim 0 \]

From now on \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^N \) and we assume

\[ |\text{adj} \, Df|, |\text{adj} \, Dg| \in L^{\frac{N}{N-1}}(\Omega) \]

\[ J_f, J_g \geq 0 \quad \text{a.e. in } \Omega \]
"Measure the distance"

\[ M \simeq 1 \implies \| Df - Dg \| \simeq 0 \]

From now on \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^N \) and we assume

\[ |\mathrm{adj} \, Df|, |\mathrm{adj} \, Dg| \in L^{\frac{N}{N-1}}(\Omega) \]

\[ J_f, J_g \geq 0 \quad \text{a.e. in } \Omega \]
Proposition 3.3

Let $f, g \in W^{1,N-1}(\Omega, \mathbb{R}^N)$. Assume that $f$ and $g$ satisfy relations (6) and (7) respectively

\[
\left( \int_{B_r(x_0)} J_f(x) \, dx \right)^{\frac{N-1}{N}} = \int_{\partial B_r(x_0)} |\text{adj} \, Df| \, d\mathcal{H}^{N-1} \tag{6}
\]

\[
\int_{\partial B_r(x_0)} |\text{adj} \, Dg| \, d\mathcal{H}^{N-1} \leq M \left( \int_{B_r(x_0)} J_g(x) \, dx \right)^{\frac{N-1}{N}} \tag{7}
\]

and $f = g$ on $\partial \Omega$.

Then $f^i$ and $g^i$, $i \in \{1, \ldots, N\}$, solve the problem

\[
\begin{cases}
\text{div} \, A(x, \nabla g^i) = \text{div} \left( |\nabla f^i|^{N-2} \nabla f^i \right) \quad \text{in } \Omega, \\
g^i = f^i \quad \text{on } \partial \Omega,
\end{cases} \tag{8}
\]

where
Proposition 3.3

Let \( f, g \in W^{1,N-1}(\Omega, \mathbb{R}^N) \). Assume that \( f \) and \( g \) satisfy relations (6) and (7) respectively

\[
\left( \int_{B_r(x_0)} J_f(x) \, dx \right)^{\frac{N-1}{N}} = \int_{\partial B_r(x_0)} \| \text{adj} \, Df \| \, d\mathcal{H}^{N-1}
\]

and

\[
\int_{\partial B_r(x_0)} \| \text{adj} \, Dg \| \, d\mathcal{H}^{N-1} \leq M \left( \int_{B_r(x_0)} J_g(x) \, dx \right)^{\frac{N-1}{N}}
\]

and \( f = g \) on \( \partial \Omega \).

Then \( f^i \) and \( g^i, i \in \{1, \ldots, N\} \), solve the problem

\[
\begin{cases}
\text{div} \, \mathcal{A}(x, \nabla g^i) = \text{div} \left( |\nabla f^i|^{N-2} \nabla f^i \right) \quad \text{in } \Omega, \\
g^i = f^i \quad \text{on } \partial \Omega,
\end{cases}
\]

where
\[ \mathcal{A}(x, \xi) = \langle A_g(x)\xi, \xi \rangle^{\frac{N-2}{2}} A_g(x)\xi \]

for a.e. \( x \in \Omega \), for every \( \xi \in \mathbb{R}^N \) and

\[
A_g(x) = \begin{cases} 
\frac{(\text{Adj } Dg) (\text{Adj } Dg)^t}{J_g(x)^2(N-2)/N} & \text{if } J_g(x) > 0, \\
\text{Id} & \text{if } J_g(x) = 0,
\end{cases}
\]

(9)

\( A(x, g) \equiv A_g(x) \) is the inverse of the so called "distortion tensor" of \( g \), and satisfies

\[
\mu |\xi|^2 \leq \langle A_g(x)\xi, \xi \rangle \leq \nu |\xi|^2,
\]

\[
\mu = \mu(N, M) \quad \nu = \nu(N, M)
\]

for a.e. \( x \in \Omega \), for every \( \xi \in \mathbb{R}^N \).
\[ A(x, \xi) = \langle A_g(x)\xi, \xi \rangle^{\frac{N-2}{2}} A_g(x)\xi \]

for a.e. \( x \in \Omega \), for every \( \xi \in \mathbb{R}^N \) and

\[
A_g(x) = \begin{cases} 
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\]

\[
\mu = \mu(N, M) \quad \nu = \nu(N, M)
\]

for a.e. \( x \in \Omega \), for every \( \xi \in \mathbb{R}^N \).
(8) holds in the "distributional" sense, i.e.

\[ \int_{\Omega} \langle A(x, \nabla g^i), \nabla \varphi \rangle dx = \int_{\Omega} \langle |\nabla f^i|^{N-2} \nabla f^i, \nabla \varphi \rangle dx \]

for every \( \varphi \in C_0^\infty(\Omega) \).
Then by Proposition 2.1 we get the following result

**Proposition 3.4**

Let $f, g \in W^{1,N}(\Omega, \mathbb{R}^N)$. Assume that (6) and (7) hold and $f = g$ on $\partial \Omega$. Then

$$\|\nabla f^i - \nabla g^i\|_N \leq C \left( M^{N-1} - 1 \right)^{\frac{1}{N-1}} \|\nabla f^i\|_N$$

where $C = C(N)$, $i \in \{1, \ldots, N\}$. 
Then by Proposition 2.1 we get the following result

**Proposition 3.4**

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where $C = C(N)$, $i \in \{1, \ldots, N\}$. 
For $M = 1$, we get the following uniqueness result

**Corollary 1**

Let $f, g \in W^{1,N}(\Omega, \mathbb{R}^N)$ satisfy (6). If $f = g$ on $\partial\Omega$ then $f = g$ in $\Omega$.

**Remark** The result is sharp. Indeed, for $g \equiv Id$ it is possible to find a mapping $f$ s.t. $f = Id$ on $\partial\Omega$, equality (6) holds for some $x_0 \in \Omega$ and some radius $r$, but $f$ is not the identity map.
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**Corollary 1**

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Theorem 3.2

Let \( f, g \in W^{1,1}(\Omega, \mathbb{R}^N) \). Assume that

\[
Df, Dg \in L^N \log^{-\alpha} L(\Omega, \mathbb{R}^{N \times N}) \quad 0 < \alpha < \frac{N}{N - 2}
\]

and satisfy (6) and (7), respectively, \( f = g \) on \( \partial \Omega \), then

\[
\| \nabla f^i - \nabla g^i \|_{L^N \log^{-\alpha} L(\Omega)} \leq CM^{1-\gamma} (M^{N-1} - 1)^{\frac{\gamma}{N-1}} \| \nabla f^i \|_{L^N \log^{-\alpha} L(\Omega)}
\]

where \( C = C(N, \alpha) > 0 \) and \( \gamma = 1 - \alpha \frac{N-2}{N}, \, i \in \{1, \ldots, N\} \).
An estimate in a parabolic problem

Let us consider the problem

\[
\begin{cases}
  u_t - \text{div}\mathcal{A}(x, t) \nabla u = -\text{div} h & \text{in } \Omega \times (0, +\infty) \\
  u = 0 & \text{on } \partial\Omega \times (0, +\infty) \\
  u(\cdot, 0) = u_0 & \text{in } \Omega
\end{cases}
\]

where \( \mathcal{A}(x, t) \) is a Carathéodory function satisfying, for a.e. \((x, t) \in \Omega \times (0, +\infty) \) and for all \( \xi \in \mathbb{R}^N \)

\[
\langle \mathcal{A}(x, t)\xi, \xi \rangle \geq \alpha |\xi|^2 \quad \alpha > 0
\]

\(|\mathcal{A}(x, t)| \leq M. \]

We assume that \( h = h(x) \in L^2(\Omega, \mathbb{R}^N) \)
Now we consider the Dirichlet problem

\[
(b) \quad \begin{cases}
\quad \text{div}\mathcal{B}(x)\nabla v = \text{div}\, h & \text{in } \Omega \\
\quad v = 0 & \text{on } \partial \Omega
\end{cases}
\]

where \( \mathcal{B}(x) \) is a measurable function s.t. for a.e. \( x \in \Omega \) and for all \( \xi \in \mathbb{R}^N \)

\[
\langle \mathcal{B}(x)\xi, \xi \rangle \geq \beta |\xi|^2, \quad |\mathcal{B}(x)| \leq M_1.
\]

Let us define

\[
H(t) = \|A(x, t) - \mathcal{B}(x)\|_{L^\infty(\Omega)}.
\]
Theorem

Let \( u \in L^2_{\text{loc}}((0, +\infty); H^1_0(\Omega)) \) and \( v \in H^1_0(\Omega) \) solutions of problems (a) and (b) respectively. If \( H(t) \in L^2_{\text{loc}}(0, +\infty) \), then for every \( t \in (0, +\infty) \)

\[
\| u(\cdot, t) - v \|_{L^2(\Omega)}^2 \leq \left[ \| u_0 - v \|_{L^2(\Omega)}^2 + \| \nabla v \|_{L^2(\Omega)}^2 \alpha^{-1} \right. \\
\left. \int_0^t H^2(s) e^{cs} \, ds \right] e^{-ct}
\]

where \( c = \frac{\alpha}{c_P} \) and \( c_P \) is the constant in Poincaré-inequality.
Remark 1

If $H \in L^2(0, +\infty)$ then

$$
\|u(\cdot, t) - v\|_{L^2(\Omega)}^2 \leq \|u_0 - v\|_{L^2(\Omega)}^2 e^{-ct} + \|
abla v\|_{L^2(\Omega)}^2 \alpha^{-1} \int_0^{+\infty} H^2(s) ds
$$

Remark 2

If

$$
\int_0^{+\infty} H^2(s) e^{cs} ds < \infty,
$$

then

$$
\lim_{t \to \infty} \|u(\cdot, t) - v\|_{L^2(\Omega)} = 0
$$
THANK YOU FOR YOUR ATTENTION!